PERGAMON

International Journal of Solids and Structures 39 (2002) 4421–4436

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

In-plane perturbation of the tunnel-crack under shear loading I: bifurcation and stability of the straight configuration of the front

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Received 31 October 2001; received in revised form 4 April 2002

Abstract

One considers a planar tunnel-crack embedded in an infinite isotropic brittle solid and loaded in mode 2 + 3 through some uniform shear remote loading. The crack front is slightly perturbed within the crack plane, from its rectilinear configuration. Part I of this work investigates the two following questions: Is there a wavy "bifurcated" configuration of the front for which the energy release rate is uniform along it? Will any given perturbation decay or grow during propagation? To address these problems, the distribution of the stress intensity factors (SIF) and the energy release rate along the perturbed front is derived using Bueckner–Rice's weight function theory. A "critical" sinusoidal bifurcated configuration of the front is found; both its wavelength and the "phase difference" between the fore and rear parts of the front depend upon the ratio of the initial (prior to perturbation of the front) mode 2 and 3 SIF. Also, it is shown that the straight configuration of the front is stable versus perturbations with wavelength smaller than the critical one but unstable versus perturbations with wavelength larger than it. This conclusion is similar to those derived by Gao and Rice and the authors for analogous problems.

Keywords: Tunnel-crack; Shear loading; Mode 2; Mode 3; Crack front; In-plane perturbation; Bifurcation; Stability; Bueckner–Rice theory; Weight functions

1. Introduction

Consider a plane crack with arbitrary contour $\mathscr{F}$, embedded in an arbitrary isotropic elastic body Ω. Let M denote the generic point of $\mathscr{F}$. If the crack advances, under constant loading, by a small distance $\delta a(M)$ within the plane in the direction perpendicular to the front $\mathscr{F}$, the variations $\delta K_m(M)$, $m = 1, 2, 3$ of the stress intensity factors (SIF) at point M are given, to first order in the perturbation, by the following formulae:

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PII: S0020-7683(02)00349-9

$$\delta K_m(M) = [\delta K_m(M)]_{\delta a(M') \equiv \delta a(M)} + N_{mn} K_n(M) \frac{\mathrm{d}\delta a(M)}{\mathrm{d}s} + \mathrm{PV} \int_{\mathscr{F}} Z_{mn}(\Omega; M, M') K_n(M') [\delta a(M') - \delta a(M)] \, \mathrm{d}s', \tag{1}$$

where s, s' denote the curvilinear abscissae along the front of points M and M' respectively, and Einstein's implicit summation convention is employed for the index $n = 1, 2, 3$. In these equations, the $K_n(M)$ are the initial SIF (prior to perturbation of the crack front). The $[\delta K_m(M)]_{\delta a(M') \equiv \delta a(M)}$ are the values of the $\delta K_m(M)$ for a uniform advance equal to $\delta a(M)$ ($\delta a(M') \equiv \delta a(M), \forall M' \in \mathscr{F}$). The N_{mn} are the components of a *universal* (geometry-independent) operator. They are given by (other components being zero):

$$N_{23} = -\frac{2}{2 - \nu}, \quad N_{32} = \frac{2(1 - \nu)}{2 - \nu}, \tag{2}$$

where ν denotes Poisson's ratio. The $Z_{mn}(\Omega; M, M')$ are the components of a *non-universal* (geometry dependent, whence the argument "Ω") operator $\mathbf{Z}$ called the *fundamental kernel* in the sequel, since it appears as the kernel of the principal value (PV) integral. Some general properties of these functions are as follows:

$$\begin{aligned} &Z_{mn}(\Omega; M, M') = Z_{nm}(\Omega; M', M), \ (m, n) = (1, 1); (2, 2); (3, 3); (1, 2), \\ &Z_{3n}(\Omega; M, M') = (1 - \nu) Z_{n3}(\Omega; M', M), \ n = 1; 2, \end{aligned} \tag{3}$$

$$\begin{aligned} &\lim_{M' \to M} Z_{11}(\Omega; M, M') D^2(M, M') = \frac{1}{2\pi}, \\ &\lim_{M' \to M} Z_{22}(\Omega; M, M') D^2(M, M') = \frac{2 - 3\nu}{2\pi(2 - \nu)}, \\ &\lim_{M' \to M} Z_{33}(\Omega; M, M') D^2(M, M') = \frac{2 + \nu}{2\pi(2 - \nu)}, \\ &\lim_{M' \to M} Z_{mn}(\Omega; M, M') D^2(M, M') = 0, \quad m \neq n, \end{aligned} \tag{4}$$

where $D(M, M')$ denotes the Cartesian distance between the points M and M'. Note that Eq. (4) warrant that the integral in Eq. (1) makes sense as a Cauchy principal value.

Eq. (1) was first established by Gao and Rice for various particular cases: the half-plane crack in mode 1 (Rice, 1985) and $1 + 2 + 3$ (Gao and Rice, 1986), the circular connection in mode 1 (Gao and Rice, 1987a), the penny-shaped crack in mode 1 (Gao and Rice, 1987b) and $1 + 2 + 3$ (Gao, 1988). In all these papers, the fundamental kernel did not appear in the generic name $\mathbf{Z}$ but in an explicit form depending on the configuration studied. Eq. (1) was then extended by Rice (1989) and Nazarov (1989) to arbitrary planar crack shapes in mode 1, and finally by Mouchrif (1994) and Leblond et al. (1999) to cracks of completely arbitrary, non-planar shapes including possible kink angles and arbitrary combinations of modes. In these more general cases, since the fundamental kernel depends on the geometry which was supposed to be more or less arbitrary, it appeared under a generic form; $\mathbf{Z}$ is the notation used in Leblond et al. (1999). The values of the N_{mn} given by (2) can be deduced from the works of Rice (1985), Gao and Rice (1986, 1987a,b) and Gao (1988). Finally, properties (3) and (4) were proved for arbitrary plane cracks loaded in pure mode 1 by both Rice (1989) and Nazarov (1989), and for arbitrary curved crack geometries and mixed mode conditions by Mouchrif (1994) and Leblond et al. (1999). All these works heavily relied on the use of Bueckner–Rice's weight-function theory (Bueckner, 1970; Rice, 1972; Rice, 1985).

In all the above-mentioned special cases, the problems of configurational bifurcation and stability of the crack front during in-plane propagation, under uniform remote loading, could be addressed by using the explicit expression of $\mathbf{Z}$ in Eq. (1) to calculate the energy release rate along a slightly perturbed front. The *bifurcation problem* was the following one: is there, in addition to the trivial, initial (straight or circular) configuration of the crack front, some non-trivial, wavy configuration for which the energy release rate is still uniform? The *stability problem* was as follows: if the crack front is slightly perturbed within the crack plane, will the perturbation decay or increase as propagation proceeds? This issue could be addressed by considering that the perturbation decayed if the energy release rate was lowest at the most advanced parts of the crack front, and that it grew if the opposite held true.

For half-plane cracks and internal circular cracks under mixed mode loadings, and for circular connections under mode 1 loading, Rice (1985), Gao and Rice (1986, 1987a,b) and Gao (1988) have shown that there is a sinusoidal bifurcated configuration of "critical" wavelength λ_c, and that stability prevails for perturbations of wavelength smaller than λ_c and instability for wavelengths larger than it. An analogous result has been established for interface half-plane cracks by Lazarus and Leblond (1998). However, for half-plane cracks, as pointed out by Gao and Rice (1986) because of the lack of a characteristic lengthscale in the problem, the somewhat deceiving conclusion is only that "planar crack growth should be configurationally stable to perturbations involving wavelengths that are small compared to overall body or crack dimensions". Leblond et al. (1996) have introduced a characteristic lengthscale by studying the *tunnel-crack* under mode 1 loading. They have shown that the critical wavelength λ_c is a characteristic multiple of the crack width and that the critical bifurcated configuration is symmetric with respect to the middle axis of the crack.

The aim of Part I of this work is to consider the same bifurcation and stability problems for the tunnel-crack as Leblond et al. (1996), but for mixed mode $(2+3)$ shear loadings. Propagation is assumed to be coplanar; this is reasonable provided that the crack is channeled along a planar surface of low fracture resistance, which can be the case for instance for a geological fault. Also, propagation is considered to be governed by the local energy release rate, the critical value of which is assumed to be independent of mode combination. Again, this is reasonable (Rice, private communication) for coplanar propagation along a weak surface, since energy dissipation occurs through the same physical mechanisms (shear and friction) in both modes 2 and 3.

Bifurcation and stability issues of course depend on the geometry considered. Therefore some general properties of the operator $\mathbf{Z}$ for a tunnel-crack, are needed as a prerequisite. These properties are expounded in Section 2. They allow us to derive, in Section 3, an expression of the variation of the energy release rate due to a small wavy perturbation of the crack front that forms the basis of our discussion of bifurcation and stability problems. In Section 4, it is then shown that there is a critical, sinusoidal bifurcated configuration of the front. Its wavelength is a multiple of the width of the crack and depends upon the ratio of the mode 2 and 3 initial SIF (prior to perturbation of the front). Also, the bifurcated configuration is symmetric with respect to the middle axis of the crack only for initial conditions of pure mode 2 or 3; for mixed mode conditions, there is a "phase difference" between the bifurcated configurations of the fore and rear parts of the crack front. The stability issue is addressed, in Section 5, only in some simple, special cases where the extrema of the perturbation and the energy release rate coincide. It is shown that in the most interesting case, stability prevails for perturbations of wavelength smaller than the critical one.

It should be noticed that a significant part of the analysis of bifurcation and stability can be carried out using only the properties of the fundamental kernel $\mathbf{Z}$ expounded in Section 2, that is without explicitly knowing its components. However, such an explicit knowledge is of course necessary for a fully quantitative analysis. But the calculation of $\mathbf{Z}$ is long and complex; for reasons of compactness of the present paper, we shall therefore merely accept its results here and postpone its detailed presentation to Part II.

2. General properties of the fundamental kernel Z

2.1. Definitions and notations

The situation considered is depicted in Fig. 1. The crack lies on the plane $y=0$ and the fore and rear parts of the front are parallel straight lines of equation $(x=a)$ and $(x=-a)$ respectively. The position of a point M of the front is specified through the Cartesian coordinate $z^{\pm}$, where the upper index indicates

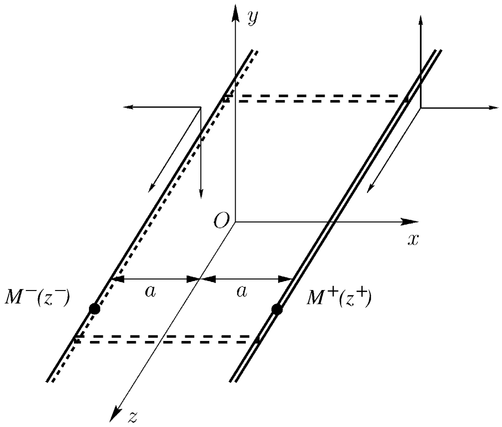

Fig. 1. Tunnel-crack of width $2a$.

whether the point considered belongs to the fore $(x = a)$ or rear part $(x = -a)$ of the front. The SIF at a point M of the front are defined with respect to the local set of axes (x, y, z) if M belongs to the line $(x = a)$ and $(-x, -y, z)$ if it belongs to the line $(x = -a)$.

The only geometric parameter in the problem is the half-width a of the crack; it follows that the influence of the argument "Ω" upon the fundamental kernel $\mathbf{Z}(\Omega; M, M')$ in fact reduces to a dependence of this operator upon a. Furthermore, the problem is invariant in the direction z of the crack front and simple dimensional considerations in Eq. (1) show that $\mathbf{Z}(\Omega; M, M') \equiv \mathbf{Z}(a; z^{\pm}, z'^{\pm})$ is positively homogeneous of degree -2 with respect to its three arguments. Combining these features with the obvious symmetry with respect to the central axis Oz, one concludes that the fundamental kernel can be written in the following form:

$$\mathbf{Z}(a; z^+, z'^+) = \mathbf{Z}(a; z^-, z'^-) \equiv \frac{\mathbf{f}((z' - z)/a)}{(z' - z)^2}, \tag{5}$$

$$\mathbf{Z}(a; z^+, z'^-) = \mathbf{Z}(a; z^-, z'^+) \equiv \frac{\mathbf{g}((z' - z)/a)}{a^2}, \tag{6}$$

where, in virtue of Eq. (4), the components of operators $\mathbf{f}$ and $\mathbf{g}$ are bounded for $z' \to z$ and verify the following properties:

$$\begin{aligned}
&\lim_{z' \to z} f_{11}((z' - z)/a) = \frac{1}{2\pi}, \\
&\lim_{z' \to z} f_{22}((z' - z)/a) = \frac{2 - 3\nu}{2\pi(2 - \nu)}, \\
&\lim_{z' \to z} f_{33}((z' - z)/a) = \frac{2 + \nu}{2\pi(2 - \nu)}, \\
&\lim_{z' \to z} f_{mn}((z' - z)/a) = 0, \quad m \neq n.
\end{aligned} \tag{7}$$

Another basic property of $\mathbf{f}$ and $\mathbf{g}$ is that:

$$f_{12} = f_{21} = f_{13} = f_{31} = g_{12} = g_{21} = g_{13} = g_{31} \equiv 0. \tag{8}$$

This is because, as is well-known, tensile and shear problems are uncoupled for a planar crack with an arbitrary contour in an infinite body; that is, if $K_2 \equiv 0$ and $K_3 \equiv 0$ (tensile problem), the variations δK_2 and δK_3 are zero when the crack propagates within its plane; and similarly if $K_1 \equiv 0$ (shear problem), $\delta K_1 \equiv 0$.

Furthermore, elementary considerations of symmetry with respect to the plane $z = 0$ show that f_{11}, f_{22}, f_{33}, g_{11}, g_{22}, g_{33} are even, and f_{23}, g_{23}, f_{32} and g_{32} are odd functions. Eqs. (3), (5) and (6) then imply that (with $u \equiv (z' - z)/a$):

$$\begin{aligned} f_{32}(u) &= (1-\nu)f_{23}(-u) = -(1-\nu)f_{23}(u), \\ g_{32}(u) &= (1-\nu)g_{23}(-u) = -(1-\nu)g_{23}(u). \end{aligned} \tag{9}$$

Therefore, the fundamental kernel $\mathbf{Z}$ is entirely determined by the eight components 11, 22, 33, 23 of the operators $\mathbf{f}$ and $\mathbf{g}$.

2.2. Expressions of the δK_m, $m = 2, 3$ in terms of $\boldsymbol{f}$ and $\mathbf{g}$

As already mentioned, for the tunnel-crack under shear loading, $\delta K_1 \equiv 0$. Furthermore, for $m = 2$, 3, with the notations (5) and (6) and because of properties (8) and (9), the fundamental Eq. (1) reads for a point $M^+(z^+)$ belonging to the line $(x = a)$:

$$\begin{aligned} \delta K_2(z^+) = {} & [\delta K_2(z^+)]_{\delta a(z'^{\pm}) \equiv \delta a(z^+)} - \frac{2}{2-\nu} K_3(z^+) \frac{\mathrm{d}\delta a}{\mathrm{d}z}(z^+) \\ & + \mathrm{PV} \int_{-\infty}^{+\infty} \left[f_{22}\left(\frac{z'-z}{a}\right) K_2(z'^+) + f_{23}\left(\frac{z'-z}{a}\right) K_3(z'^+) \right] \frac{\delta a(z'^+) - \delta a(z^+)}{(z'-z)^2} \mathrm{d}z' \\ & + \int_{-\infty}^{+\infty} \left[g_{22}\left(\frac{z'-z}{a}\right) K_2(z'^-) + g_{23}\left(\frac{z'-z}{a}\right) K_3(z'^-) \right] \frac{\delta a(z'^-) - \delta a(z^+)}{a^2} \mathrm{d}z', \end{aligned} \tag{10}$$

$$\begin{aligned} \delta K_3(z^+) = {} & [\delta K_3(z^+)]_{\delta a(z'^{\pm}) \equiv \delta a(z^+)} + \frac{2(1-\nu)}{2-\nu} K_2(z^+) \frac{\mathrm{d}\delta a}{\mathrm{d}z}(z^+) \\ & + \mathrm{PV} \int_{-\infty}^{+\infty} \left[f_{33}\left(\frac{z'-z}{a}\right) K_3(z'^+) - (1-\nu) f_{23}\left(\frac{z'-z}{a}\right) K_2(z'^+) \right] \frac{\delta a(z'^+) - \delta a(z^+)}{(z'-z)^2} \mathrm{d}z' \\ & + \int_{-\infty}^{+\infty} \left[g_{33}\left(\frac{z'-z}{a}\right) K_3(z'^-) - (1-\nu) g_{23}\left(\frac{z'-z}{a}\right) K_2(z'^-) \right] \frac{\delta a(z'^-) - \delta a(z^+)}{a^2} \mathrm{d}z'. \end{aligned} \tag{11}$$

The values of $\delta K_2(z^-)$ and $\delta K_3(z^-)$, for a point $M^-(z^-)$ belonging to the line $(x = -a)$, are given by the same expressions with the obvious substitutions $z^+ \to z^-$, $z'^{\pm} \to z'^{\mp}$.

3. Perturbation of the tunnel-crack under uniform remote shear loading

Let the tunnel-crack be subjected to some uniform remote plane (τ_p) and antiplane (τ_a) shear loading so that Cauchy stress tensor at infinity reads $\boldsymbol{\sigma}_\infty = \tau_p(\overrightarrow{e_y} \otimes \overrightarrow{e_x} + \overrightarrow{e_x} \otimes \overrightarrow{e_y}) + \tau_a(\overrightarrow{e_y} \otimes \overrightarrow{e_z} + \overrightarrow{e_z} \otimes \overrightarrow{e_y})$. Then the SIF, prior to any perturbation of the front, are given by:

$$K_1(z^\pm) = 0; \quad K_2(z^\pm) = \tau_p\sqrt{\pi a} \equiv K_2; \quad K_3(z^+) = \tau_a\sqrt{\pi a} \equiv K_3^+; \quad K_3(z^-) = -\tau_a\sqrt{\pi a} = -K_3^+ \equiv K_3^-. \tag{12}$$

The variation of the SIF will be studied for three types of perturbation of the front: translation, rotation and sinusoidal undulation (Fig. 2). The study of the translation will serve to simplify (in the case of a uniform remote loading), the expressions (10) and (11) of $\delta K_2(z^+)$ and $\delta K_3(z^+)$. The study of the rotation will allow the calculation of some integrals involving operators $\mathbf{f}$ and $\mathbf{g}$. These integrals are given here although they will be needed only in Part II, because the reasoning is similar to that for the translation. The study of the sinusoidal undulation is a necessary prerequisite for that of the bifurcation and stability problems.

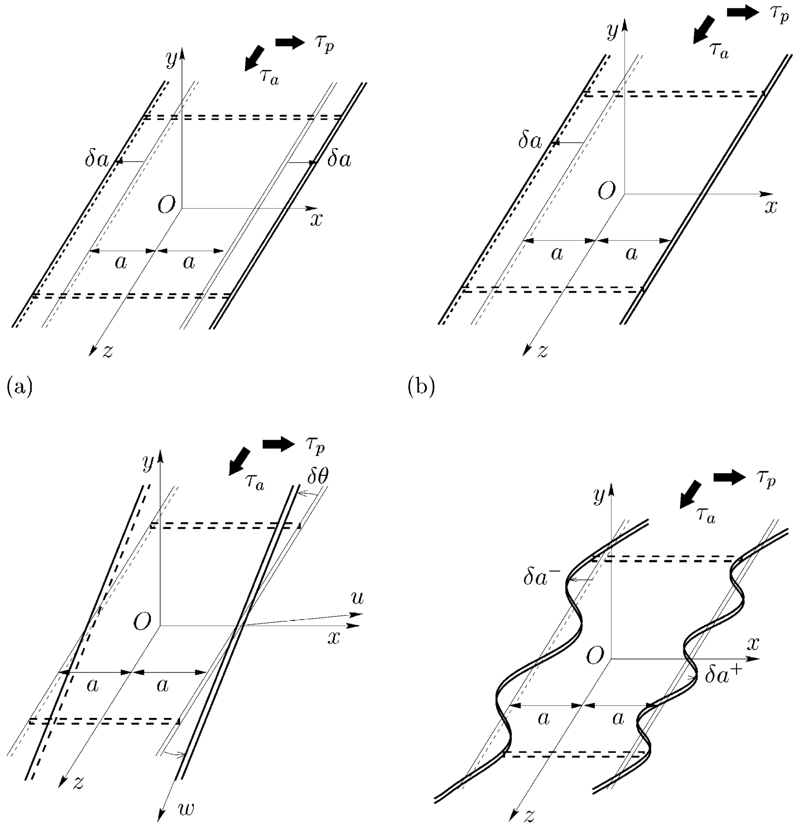

Fig. 2. In-plane perturbations of the tunnel-crack under uniform remote shear loading. (a) Translation of both parts of the front. (b) Translation of the rear part of the front. (c) Rotation about the Oy axis. (d) Wavy front.

3.1. Translation of the front

If both parts of the front move by a uniform amount δa as in Fig. 2(a), the new SIF are those of a tunnel-crack of width $2(a + \delta a)$ subjected to the remote loading $\boldsymbol{\sigma}_\infty$. Hence to first order in $\delta a/a$:

$$\begin{aligned} &[\delta K_2(z^\pm)]_{\delta a(z'^\pm)\equiv\delta a} = K_2 \frac{\delta a}{2a}, \\ &[\delta K_3(z^+)]_{\delta a(z'^\pm)\equiv\delta a} = K_3^+ \frac{\delta a}{2a}, \quad [\delta K_3(z^-)]_{\delta a(z'^\pm)\equiv\delta a} = K_3^- \frac{\delta a}{2a}. \end{aligned} \tag{13}$$

If now the sole rear part of the front moves by an amount δa as in Fig. 2(b), the new SIF are those of a tunnel-crack of width $2a + \delta a$ subjected to the remote loading $\boldsymbol{\sigma}_\infty$. Thus $\delta K_2(z^\pm) = (K_2 \delta a)/(4a)$ and $\delta K_3(z^\pm) = (K_3^\pm \delta a)/(4a)$. Eqs. (10) and (11) then yield:

$$\int_{-\infty}^{+\infty} g_{22}(u)\,\mathrm{d}u = -\int_{-\infty}^{+\infty} g_{33}(u)\,\mathrm{d}u = \frac{1}{4}, \quad \int_{-\infty}^{+\infty} g_{23}(u)\,\mathrm{d}u = 0. \tag{14}$$

The last relation was obvious a priori since g_{23} is odd.

By using Eqs. (13) and (14), the relations (10) and (11) can be rewritten in the following slightly simplified form (for a uniform remote loading):

$$\begin{aligned}\delta K_2(z^+) = {} & \frac{K_2}{4}\frac{\delta a(z^+)}{a} - \frac{2}{2-\nu}K_3^+\frac{\mathrm{d}\delta a}{\mathrm{d}z}(z^+) + K_2 \mathrm{PV}\int_{-\infty}^{+\infty} f_{22}\left(\frac{z'-z}{a}\right)\frac{\delta a(z'^+)-\delta a(z^+)}{(z'-z)^2}\,\mathrm{d}z' \\ & + K_3^+\int_{-\infty}^{+\infty} f_{23}\left(\frac{z'-z}{a}\right)\frac{\delta a(z'^+)-\delta a(z^+)}{(z'-z)^2}\,\mathrm{d}z' + K_2\int_{-\infty}^{+\infty} g_{22}\left(\frac{z'-z}{a}\right)\frac{\delta a(z'^-)}{a^2}\,\mathrm{d}z' \\ & + K_3^-\int_{-\infty}^{+\infty} g_{23}\left(\frac{z'-z}{a}\right)\frac{\delta a(z'^-)}{a^2}\,\mathrm{d}z',\end{aligned} \tag{15}$$

$$\begin{aligned}\delta K_3(z^+) = {} & \frac{K_3^+}{4}\frac{\delta a(z^+)}{a} + \frac{2(1-\nu)}{2-\nu}K_2\frac{\mathrm{d}\delta a}{\mathrm{d}z}(z^+) + K_3^+ \mathrm{PV}\int_{-\infty}^{+\infty} f_{33}\left(\frac{z'-z}{a}\right)\frac{\delta a(z'^+)-\delta a(z^+)}{(z'-z)^2}\,\mathrm{d}z' \\ & - (1-\nu)K_2\int_{-\infty}^{+\infty} f_{23}\left(\frac{z'-z}{a}\right)\frac{\delta a(z'^+)-\delta a(z^+)}{(z'-z)^2}\,\mathrm{d}z' + K_3^-\int_{-\infty}^{+\infty} g_{33}\left(\frac{z'-z}{a}\right)\frac{\delta a(z'^-)}{a^2}\,\mathrm{d}z' \\ & - (1-\nu)K_2\int_{-\infty}^{+\infty} g_{23}\left(\frac{z'-z}{a}\right)\frac{\delta a(z'^-)}{a^2}\,\mathrm{d}z',\end{aligned} \tag{16}$$

where the "PV" symbols have been canceled in the integrals involving f_{23} since this function is odd.

The expressions of $\delta K_2(z^-)$ and $\delta K_3(z^-)$ are similar with the substitutions $z^+ \to z^-$, $z'^{\pm} \to z'^{\mp}$, $K_3^{\pm} \to K_3^{\mp}$.

3.2. Rotation of the front

Let us suppose that the perturbation is a rotation of both parts of the front about $\overrightarrow{e_y}$ as in Fig. 2(c) so that $\delta a(z^+) = \delta\theta.z$ and $\delta a(z^-) = -\delta\theta.z$, $|\delta\theta| \ll 1$. Then to first order in $\delta\theta$, the new SIF are those of a tunnel-crack of width $2a$ subjected to the uniform remote loading $(\tau_p - \delta\theta.\tau_a)(\overrightarrow{e_y}\otimes\overrightarrow{e_u} + \overrightarrow{e_u}\otimes\overrightarrow{e_y}) + (\tau_a + \delta\theta.\tau_p)(\overrightarrow{e_y}\otimes\overrightarrow{e_w} + \overrightarrow{e_w}\otimes\overrightarrow{e_y})$ where $\overrightarrow{e_u}$ and $\overrightarrow{e_w}$ are defined in Fig. 2(c). Thus $\delta K_2(z^\pm) = -\delta\theta.K_3^+$ and $\delta K_3(z^\pm) = \pm\delta\theta.K_2$. Relations (15) and (16) then yield a system of two equations in the two unknown integrals $\int_{-\infty}^{+\infty} f_{23}(u)(\mathrm{d}u/u)$ and $\int_{-\infty}^{+\infty} u g_{23}(u)\,\mathrm{d}u$, the solution of which reads:

$$\int_{-\infty}^{+\infty} f_{23}(u)\frac{\mathrm{d}u}{u} = -\frac{\nu^2}{2(2-\nu)(1-\nu)}, \tag{17}$$

$$\int_{-\infty}^{+\infty} u g_{23}(u)\,\mathrm{d}u = \frac{\nu}{2(1-\nu)}. \tag{18}$$

3.3. Wavy perturbation of the crack front

Let us now suppose that, as in Fig. 2(d):

$$\begin{cases}\delta a(z^+) = \alpha^+\cos(k^+z + \phi^+)\\ \delta a(z^-) = \alpha^-\cos(k^-z + \phi^-)\\ \dfrac{|\alpha^+|}{a} \sim \dfrac{|\alpha^-|}{a} \ll 1.\end{cases} \tag{19}$$

Then, by substituting Eq. (19) in (15) and (16) and using the notation

$$p^{\pm} \equiv k^{\pm} a, \tag{20}$$

(p^+, p^- are "reduced", dimensionless wavevectors), one finds to first order in ($\alpha^{\pm}/a$), after a lengthy but straightforward calculation:

$$\delta K_2(z^+) = \frac{\alpha^+}{a}\cos(k^+ z + \phi^+) K_2 \bar{f}_{22}(p^+) + \frac{\alpha^+}{a}\sin(k^+ z + \phi^+) K_3^+ \mathrm{i}\bar{f}_{23}(p^+) + \frac{\alpha^-}{a}\cos(k^- z + \phi^-) K_2 \hat{g}_{22}(p^-) + \frac{\alpha^-}{a}\sin(k^- z + \phi^-) K_3^- \mathrm{i}\hat{g}_{23}(p^-), \tag{21}$$

$$\delta K_3(z^+) = \frac{\alpha^+}{a}\cos(k^+ z + \phi^+) K_3^+ \bar{f}_{33}(p^+) - (1-\nu)\frac{\alpha^+}{a}\sin(k^+ z + \phi^+) K_2 \mathrm{i}\bar{f}_{23}(p^+) + \frac{\alpha^-}{a}\cos(k^- z + \phi^-) K_3^- \hat{g}_{33}(p^-) - (1-\nu)\frac{\alpha^-}{a}\sin(k^- z + \phi^-) K_2 \mathrm{i}\hat{g}_{23}(p^-), \tag{22}$$

the expressions of $\delta K_2(z^-)$ and $\delta K_3(z^-)$ being given by the same formulae with the obvious substitutions $\pm \leftrightarrow \mp$ for the superscripts of α, k, p, ϕ and K_3. In these expressions, the functions $\bar{f}_{mn}$ are defined as:

$$\begin{aligned} \bar{f}_{mn}(p) &= \frac{1}{4} + \mathrm{PV}\int_{-\infty}^{+\infty} f_{mn}(u)\frac{\mathrm{e}^{\mathrm{i}pu}-1}{u^2}\,\mathrm{d}u = \frac{1}{4} + 2\int_0^{+\infty} f_{mn}(u)\frac{\cos pu - 1}{u^2}\,\mathrm{d}u, \quad (m,n) = (2,2); (3,3), \\ \bar{f}_{23}(p) &= -\frac{2}{2-\nu}\mathrm{i}p + \mathrm{PV}\int_{-\infty}^{+\infty} f_{23}(u)\frac{\mathrm{e}^{\mathrm{i}pu}-1}{u^2}\,\mathrm{d}u = -\frac{2}{2-\nu}\mathrm{i}p + 2\mathrm{i}\int_0^{+\infty} f_{23}(u)\frac{\sin pu}{u^2}\,\mathrm{d}u, \end{aligned} \tag{23}$$

(where use has been made of the parity properties of the f_{mn}). Also, the functions $\hat{g}_{mn}$ are the Fourier transforms of the g_{mn} defined as:

$$\hat{g}_{mn}(p) = \int_{-\infty}^{+\infty} g_{mn}(u)\mathrm{e}^{\mathrm{i}pu}\,\mathrm{d}u,$$

$$\Rightarrow \begin{cases} \hat{g}_{mn}(p) = 2\displaystyle\int_0^{+\infty} g_{mn}(u)\cos pu\,\mathrm{d}u, \quad (m,n) = (2,2); (3,3) \\ \hat{g}_{23}(p) = 2\mathrm{i}\displaystyle\int_0^{+\infty} g_{23}(u)\sin pu\,\mathrm{d}u, \end{cases} \tag{24}$$

(where use has been made of the parity properties of the g_{mn}).

Notice that $\bar{f}_{22}$, $\bar{f}_{33}$, $\hat{g}_{22}$, $\hat{g}_{33}$, $\mathrm{i}\bar{f}_{23}$, $\mathrm{i}\hat{g}_{23}$ are real so that the expressions (21) and (22) of $\delta K_2(z^+)$ and $\delta K_3(z^+)$ are real in spite of the presence of the imaginary number i.

4. Study of the bifurcation problem

Any sinusoidal perturbation of the crack front may be written, for a suitable choice of the origin, in the form (19) with $\phi^+ = 0$, $\phi^- = \phi \in [-\pi, \pi)$, and α^+, α^-, p^+, $p^- > 0$. The bifurcation problem consists in looking whether there are some constants ($\alpha^+, \alpha^-, p^+, p^-, \phi$) for which the variation of energy release rate $\delta\mathcal{G}$ due to the perturbation (19) vanishes. (In fact, what is really to be investigated is $\delta\mathcal{G} - \delta\mathcal{G}_c$ where $\mathcal{G}_c$ denotes the critical value of $\mathcal{G}$; but this is equivalent to studying $\delta\mathcal{G}$ since $\mathcal{G}_c$ is assumed to be a constant, independent of mode combination). Such a set of variables (α^+, α^-, p^+, p^-, ϕ) will be called a *bifurcation mode*.

4.1. Expression of the variation of the energy release rate

Expanding Irwin's formula to first order in $(\alpha^{\pm}/a)$, one finds that the variation of the energy release rate due to the perturbation (19) is:

$$\delta\mathscr{G}(z^{\pm}) = 2\frac{1-\nu^2}{E}\left[K_2\delta K_2(z^{\pm}) + \frac{1}{1-\nu}K_3^{\pm}\delta K_3(z^{\pm})\right], \tag{25}$$

where E is Young's modulus and $\delta K_2(z^{\pm})$ and $\delta K_3(z^{\pm})$ are given by Eqs. (21) and (22). Hence, substituting 0 for ϕ^+ and ϕ for ϕ^- in these equations, one finds that

$$\begin{aligned}\delta\mathscr{G}(z^{+}) = 2\frac{1-\nu^2}{E}\frac{K_2^2}{a}\{&\alpha^+F(p^+)\cos(k^+z) + \alpha^-[G(p^-)\cos\phi + H(p^-)\sin\phi]\cos(k^-z)\\ &+\alpha^-[H(p^-)\cos\phi - G(p^-)\sin\phi]\sin(k^-z)\},\end{aligned} \tag{26}$$

$$\begin{aligned}\delta\mathscr{G}(z^{-}) = 2\frac{1-\nu^2}{E}\frac{K_2^2}{a}\{&\alpha^-F(p^-)\cos(k^-z+\phi) + \alpha^+[G(p^+)\cos\phi + H(p^+)\sin\phi]\cos(k^+z+\phi)\\ &+\alpha^+[-H(p^+)\cos\phi + G(p^+)\sin\phi]\sin(k^+z+\phi)\}.\end{aligned} \tag{27}$$

In these expressions $F(p) \equiv F(p, K_3^+/K_2)$, $G(p) \equiv G(p, K_3^+/K_2)$, $H(p) \equiv H(p, K_3^+/K_2)$ are the quantities given by:

$$\begin{aligned}F(p) &= \bar{f}_{22}(p) + \frac{1}{1-\nu}\frac{K_3^{+2}}{K_2^2}\bar{f}_{33}(p),\\ G(p) &= \hat{g}_{22}(p) - \frac{1}{1-\nu}\frac{K_3^{+2}}{K_2^2}\hat{g}_{33}(p),\\ H(p) &= -2\mathrm{i}\frac{K_3^+}{K_2}\hat{g}_{23}(p).\end{aligned} \tag{28}$$

It was noticed by Gao and Rice (1986), Gao (1988), Lazarus and Leblond (1998) that the extrema of the perturbation of the front and of the energy release rate coincide for a half-plane or a penny-shaped crack in an homogeneous body, and for an interface half-plane crack. One could therefore speculate that this was a "general property". However, since the terms proportional to $\sin(k^-z)$ and $\sin(k^+z+\phi)$ do not vanish in the expressions (26) and (27) of $\delta\mathscr{G}(z^{\pm})$, this property does not hold for the tunnel-crack.

4.2. Graphs of functions $\bar{f}_{22}$, $\bar{f}_{33}$, $\hat{g}_{22}$, $\hat{g}_{33}$, $\hat{g}_{23}$

Knowledge of the functions $\bar{f}_{22}$, $\bar{f}_{33}$, $\hat{g}_{22}$, $\hat{g}_{33}$, $\hat{g}_{23}$ now becomes necessary to pursue the discussion. For the sake of shortness of the present paper, the rather involved calculation of these functions is postponed to Part II and we shall only give here the results obtained, for the value $\nu = 0.3$ of Poisson's ratio, in the form of graphs. (see Figs. 3 [1] and 4.)

4.3. Case where $p^+ \neq p^-$

If $p^+ \neq p^-$, for $\delta\mathscr{G}(z^{\pm})$ to be zero for all z^+ and z^-, the terms proportional to $\cos(k^+z)$, $\cos(k^-z)$, $\sin(k^-z)$ in the expression (26) of $\delta\mathscr{G}(z^+)$, and those proportional to $\cos(k^-z+\phi)$, $\cos(k^+z+\phi)$, $\sin(k^+z+\phi)$ in the

[1] Since $\bar{f}_{23}$ does not appear in expressions (26) and (27) for $\delta\mathscr{G}(z^{\pm})$, this function is not given in Fig. 3.

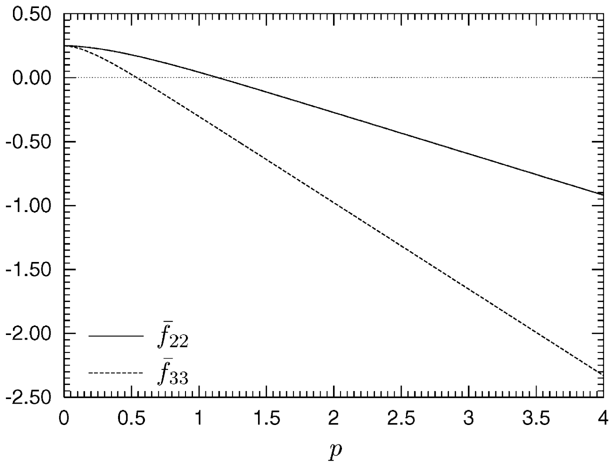

Fig. 3. Functions $\bar{f}_{mn}(p)$.

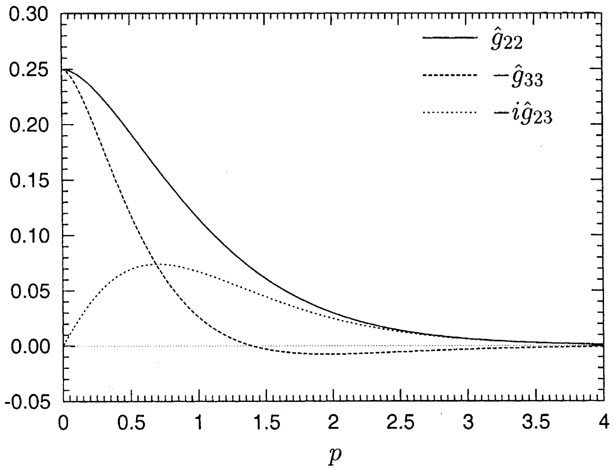

Fig. 4. Functions $\hat{g}_{mn}(p)$.

expression (27) of $\delta\mathscr{G}(z^-)$ must be individually zero. Since we are looking for non-trivial solutions, one of the coefficients α^+, α^-, say α^+, must be non-zero. The preceding conditions then implies:

$$F(p^+) = 0; \quad G(p^+)\cos\phi + H(p^+)\sin\phi = 0; \quad H(p^+)\cos\phi - G(p^+)\sin\phi = 0$$

and thus $F(p^+) = G(p^+) = H(p^+) = 0$. Now it is clear from definitions (28) and Figs. 3 and 4 that $F(p^+)$ only vanishes for some $p^+ \neq 0$ whereas $H(p^+)$ only vanishes for $p^+ = 0$. Thus these conditions cannot be satisfied for a single p^+. Hence:

There is no bifurcation mode with $p^+ \neq p^-$. (29)

4.4. Case where $p^+ = p^- \equiv p$

If $p^+ = p^- \equiv p$, for $\delta\mathscr{G}(z^\pm)$ to be zero for all z^+ and z^-, the terms proportional to $\cos(kz)$ and $\sin(kz)$ in the expression of $\delta\mathscr{G}(z^+)$, and those proportional to $\cos(kz+\phi)$ and $\sin(kz+\phi)$ in the expression of $\delta\mathscr{G}(z^-)$ must be zero. This implies that:

$$\begin{cases} \alpha^+ F(p) + \alpha^-(G(p)\cos\phi + H(p)\sin\phi) = 0 \\ \alpha^- F(p) + \alpha^+(G(p)\cos\phi + H(p)\sin\phi) = 0 \\ H(p)\cos\phi - G(p)\sin\phi = 0. \end{cases}$$

The first two equations imply that $\alpha^+/\alpha^- = \alpha^-/\alpha^+ \Rightarrow \alpha^+ = \pm\alpha^-$. Since we have chosen α^+ and α^- to be positive:

$$\begin{cases} \alpha^+ = \alpha^- \equiv \alpha \neq 0 \\ F(p) + G(p)\cos\phi + H(p)\sin\phi = 0 \\ H(p)\cos\phi - G(p)\sin\phi = 0. \end{cases} \tag{30}$$

Using second and third equations of (30), one gets $\cos\phi = -F(p)G(p)/(G^2(p) + H^2(p))$, $\sin\phi = -F(p)H(p)/(G^2(p) + H^2(p))$. Use of the relation $\cos^2\phi + \sin^2\phi = 1$ then yields

$$\begin{cases} F(p) = \pm\sqrt{G^2(p) + H^2(p)} \\ \cos\phi = -G(p)/F(p), \quad \sin\phi = -H(p)/F(p). \end{cases} \tag{31}$$

For a given ratio K_3^+/K_2, first relation of (31) is an equation on p the solution of which represents the "critical reduced wavevector". It can be solved numerically for each value of K_3^+/K_2 using values of the functions $\bar{f}_{mn}$ and $\hat{g}_{mn}$ given in Figs. 3 and 4. The second equation of Eq. (31) then define the corresponding "critical phase difference" between the configurations of the fore and rear parts of the front.

Since $\bar{f}_{22}(0) = \bar{f}_{33}(0) = \hat{g}_{22}(0) = -\hat{g}_{33}(0) = 1/4$ and $\hat{g}_{23}(0) = 0$ (see Eqs. (14), (23) and (24)), $F(0) = G(0) = (1 + (1/(1-\nu))(K_3^{+2}/K_2^2))/4$ and $H(0) = 0$ (see Eq. (28)). Therefore, if one chooses the sign + in first equation of (31), one finds that $p = 0$, $\phi = -\pi$ is a solution. This is a trivial bifurcation mode which merely corresponds to some translation of the crack in the x-direction. One can check numerically that this is the only one for the choice of the sign + in first equation of (31).

However, if one chooses the sign −, the resolution gives another unique, non-zero solution p_c and a corresponding angle ϕ_c, which define a non-trivial bifurcation mode. Hence there is a single such mode defined by the following equations:

$$\begin{cases} F(p_c) = -\sqrt{G^2(p_c) + H^2(p_c)} \\ \cos\phi_c = -G(p_c)/F(p_c), \sin\phi_c = -H(p_c)/F(p_c). \end{cases} \tag{32}$$

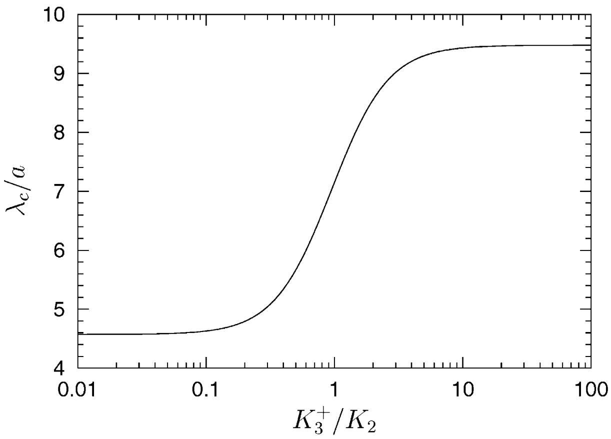

Fig. 5. Critical reduced wavelength versus the ratio of the initial SIF.

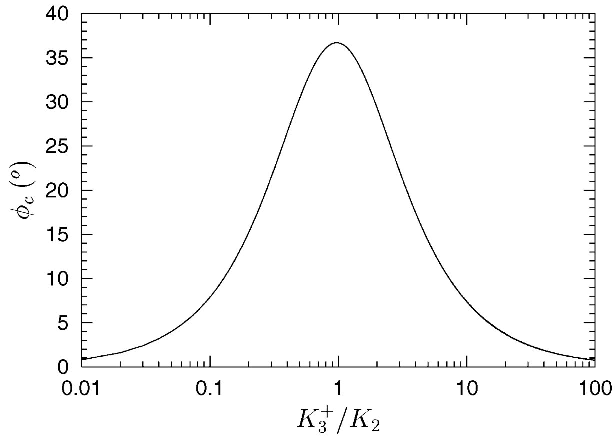

Fig. 6. Critical phase difference versus the ratio of the initial SIF.

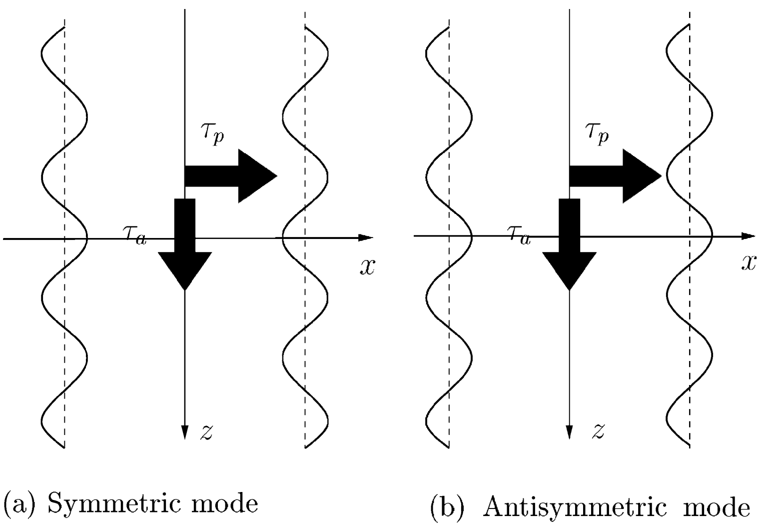

Fig. 7. Symmetric and antisymmetric modes.

Figs. 5 and 6 [2] represent the "critical reduced wavelength" $\lambda_c/a \equiv 2\pi/p_c$ and the critical phase difference ϕ_c of the bifurcated mode as functions of K_3^+/K_2, for $v = 0.3$. K_3^+/K_2 is assumed here to be positive; it is obvious that if it changes sign, λ_c remains unchanged and ϕ_c changes sign. One sees that the critical wavelength is larger in pure mode 3 than in pure mode 2. Also, the critical phase difference vanishes in pure mode 2 and mode 3, that is, the bifurcated configuration becomes symmetric with respect to the middle axis *Oz* of the crack in these cases (see Fig. 7(a)). It is recalled that the bifurcation mode was also found to be symmetric for a pure mode 1 loading (Leblond et al., 1996). Moreover $\phi_c \in (-\pi/2, \pi/2)$, hence the bifurcated mode is always closer to a symmetric configuration ($\phi = 0$, Fig. 7(a)) than to an antisymmetric one ($\phi = -\pi$, Fig. 7(b)).

[2] Note that, in spite of appearances, the curve in Fig. 6 is *not* symmetric with respect to the vertical line $K_3^+/K_2 = 1$; for instance $\phi_c \simeq 7.89^\circ$ for $K_3^+/K_2 = 0.1$ and $\phi_c \simeq 7.38^\circ$ for $K_3^+/K_2 = 10$.

5. Study of the stability problem

The question here is as follows: if the crack front is slightly perturbed within the crack plane, will the perturbation increase (*instability*) or decay (*stability*) in time? Equivalently, will the crack front depart more and more from straightness or tend to again become straight? But since not only the *amplitude* but also the *shape* of the perturbation change during propagation, the very notions of "increase" and "decay" of the perturbation are ambiguous and prone to problems of definition, so that the stability issue is complex.

In fact, we shall deal with it only in a special case for which the discussion becomes very easy and in line with previous ones of Gao and Rice cited above. This case corresponds to wavy perturbations with $p^+ = p^- \equiv p$, $\alpha^+ = \alpha^- \equiv \alpha$ and ϕ given by third equation of (30). Indeed the terms proportional to $\sin(kz)$ and $\sin(kz+\phi)$ in the expressions (26) and (27) of $\delta\mathscr{G}(z^+)$ and $\delta\mathscr{G}(z^-)$ then vanish so that the extrema of $\delta\mathscr{G}(z^+)$ coincide with those of $\delta a(z^+)$, and similarly for those of $\delta\mathscr{G}(z^-)$ and $\delta a(z^-)$. One then simply gets stability if the maxima of $\delta\mathscr{G}(z^+)$ and $\delta\mathscr{G}(z^-)$ correspond to the minima of $\delta a(z^+)$ and $\delta a(z^-)$, and instability if they correspond to the maxima of $\delta a(z^+)$ and $\delta a(z^-)$. This holds true whatever the propagation law governed by the energy release rate provided that it is independent of mode combination, and in particular for brittle fracture governed by the criterion $\mathscr{G} = \mathscr{G}_c$ if $\mathscr{G}_c$ is independent of K_3^+/K_2.

Stability then prevails if the cofactors of $\cos(kz)$ and $\cos(kz+\phi)$ in the expressions of $\delta\mathscr{G}(z^+)$ and $\delta\mathscr{G}(z^-)$ are negative [3]:

$$\text{Stability} \iff S \equiv F + G\cos\phi + H\sin\phi < 0, \quad \tan\phi = H/G. \tag{33}$$

Now,

$$\begin{aligned}\tan\phi = H/G &\Rightarrow (\cos\phi, \sin\phi) = \pm(G,H)/\sqrt{G^2+H^2}\\ &\Rightarrow F + G\cos\phi + H\sin\phi = F \pm \sqrt{G^2+H^2}.\end{aligned}$$

Therefore the stability condition (33) may be written as follows:

$$\text{Stability} \iff \begin{cases} S \equiv F + \sqrt{G^2+H^2} < 0 \text{ and } (\cos\phi, \sin\phi) = \dfrac{(G,H)}{\sqrt{G^2+H^2}} \\ \text{or} \\ S \equiv F - \sqrt{G^2+H^2} < 0 \text{ and } (\cos\phi, \sin\phi) = -\dfrac{(G,H)}{\sqrt{G^2+H^2}}. \end{cases} \tag{34}$$

Thus we should distinguish between the cases $(\cos\phi, \sin\phi) = (G,H)/\sqrt{G^2+H^2}$ and $(\cos\phi, \sin\phi) = -(G,H)/\sqrt{G^2+H^2}$:

- The more interesting case corresponds to $(\cos\phi, \sin\phi) = (G,H)/\sqrt{G^2+H^2}$. Then, for each ratio K_3^+/K_2, using the values of the functions $\bar{f}_{mn}$ and $\hat{g}_{mn}$ given in Figs. 3 and 4, one can check that S is positive for $p < p_c$ and negative for $p > p_c$: for instance, for $p = 0$, $F = G = (1 + (1/(1-\nu))(K_3^{+2}/K_2^2))/4 > 0$ and $H = 0$ (see above) so that $S = F + G > 0$, and for $p \to +\infty$, $F \to -\infty$, $G \to 0$, $H \to 0$, so that $S \sim F < 0$; also for $p = p_c$, $S = 0$ by first equation of (32). Hence *stability prevails for* $p > p_c$.
- In the less interesting case where $(\cos\phi, \sin\phi) = -(G,H)/\sqrt{G^2+H^2}$, S is negative for all ratios K_3^+/K_2 and values of p; for instance, for $p = 0$, $S = F - G = 0$, for $p \to +\infty$, $S \sim F < 0$, and for $p = p_c$, $S = -2\sqrt{G^2+H^2} < 0$ by first equation of (32). Thus *stability always prevails*.

[3] In these equations and the sequel, indications of dependence of functions F, G, H and S upon p are left out for the sake of simplicity.

Now consider an unstable configuration, having thus $p < p_c$ and $(\cos\phi, \sin\phi) = (G, H)/\sqrt{G^2 + H^2}$. Then $\lambda/a \equiv 2\pi/p > \lambda_c/a \equiv 2\pi/p_c > 4.5$ (see Fig. 5) $\Rightarrow p < 1.4 \Rightarrow -\hat{g}_{33} > 0$ (see Fig. 4) $\Rightarrow G > 0$ (see second equation of (28) and Fig. 4) $\Rightarrow \cos\phi > 0 \Rightarrow \phi \in (-\pi/2, \pi/2)$. On the other hand, consider a (stable) configuration having also $p < p_c$ but $(\cos\phi, \sin\phi) = -(G, H)/\sqrt{G^2 + H^2}$. Then, by the same reasoning, $\cos\phi < 0 \Rightarrow \phi \in [-\pi, -\pi/2] \cup (\pi/2, \pi)$. Thus, among configurations having $p < p_c$, unstable ones are characterized by the fact that they have $\phi \in (-\pi/2, \pi/2)$. Since configurations having $p > p_c$ are stable, *unstable configurations are completely characterized, among all possible ones, by the fact that they have both* $p < p_c$ *and* $\phi \in (-\pi/2, \pi/2)$; that is, their wavelength is larger than the critical one ($\lambda > \lambda_c$) and they are closer to a symmetric configuration ($\phi = 0$, Fig. 7(a)) than to an antisymmetric one ($\phi = -\pi$, Fig. 7(b)). In more discursive terms:

- If the configuration of the front is closer to a symmetric one than to an antisymmetric one, *stability prevails for wavelengths smaller than the critical value and instability for wavelengths greater than it.* This finding is similar to those of Leblond et al. (1996) in pure mode 1, Gao and Rice (1986) and Gao (1988) for half-plane and penny-shaped cracks in mode 1 and 2 + 3, and Lazarus and Leblond (1998) for interface half-plane cracks in mode 1 + 2 + 3.
- If the configuration of the front is closer to an antisymmetric one than to a symmetric one, *stability prevails for all wavelengths*.

Two final remarks are in order. First, what was considered above was (just like in previous works of Gao and Rice cited above) the question of stability versus perturbations of *fixed, prescribed* wavelength. One may also raise the question of stability versus *arbitrary* perturbations. In this respect, the straight configuration of the front is inherently unstable, since whatever the crack width, any perturbation having $p^+ = p^-$, $\alpha^+ = \alpha^-$, $(\cos\phi, \sin\phi) = (G, H)/\sqrt{G^2 + H^2}$ and $\lambda > \lambda_c$ is bound to develop in time, as discussed above.

Second, in the case of pure mode 1, for the same geometrical configuration, Leblond et al. (1996) have studied the stability problem without any restrictions on $\alpha^{\pm}$ and ϕ, thus in the absence of coincidence of the extrema of $\delta a(z^{\pm})$ and $\delta\mathcal{G}(z^{\pm})$. It is probably possible to extend their approach to mixed mode 2 + 3. But the study is then much more involved, and furthermore feasible only for fatigue or subcritical propagation laws but not for brittle fracture. These were the two reasons for considering only a simple special case here, leaving the extension of Leblond et al. (1996)'s study to mode 2 + 3 for future work.

6. Conclusions and perspectives

It has been shown that the only non-trivial bifurcated mode has the same amplitude and wavelength λ_c on both parts of the front. However, for mixed mode 2 + 3 loading conditions, there is a "phase difference" ϕ_c between the configurations of the two parts of the front depending upon the ratio of the initial mode 2 and 3 SIF. In contrast, in pure mode 2 or 3, the bifurcated mode is symmetric with respect to the central axis Oz of the crack.

The stability problem of the rectilinear configuration of the crack front has been studied only for some simple, special wavy perturbations for which the extrema of the perturbation and the energy release rate coincide. It has been shown that instability prevails for wavelengths larger than the critical one λ_c if the configuration of the front is close to a symmetric one and stability in the other cases, in particular if the configuration of the front is close to an antisymmetric one.

The wavy bifurcated configuration of the front may recall, although the problem is not of same nature, the telephone cord blisters appearing in thin films, observed for instance by Gille and Rau (1984) or

Thouless (1993). But the fore and rear parts of the front of the blister are in an antisymmetric mode and cannot therefore correspond to our bifurcated mode or instability domain.

This work is liable to extensions along three lines:

(1) The first one would be to discuss stability versus wavy perturbations of fixed wavelength without any restrictive condition ensuring coincidence of their extrema and those of the energy release rate. This seems feasible through extension of the work of Leblond et al. (1996) pertaining to the same geometric configuration but pure mode 1 conditions, to general loading conditions. However, this implies dropping the brittle-type criterion $\mathscr{G} = \mathscr{G}_c$ and adopting some subcritical growth or fatigue propagation law instead.

(2) The second one would be to consider the more general stability problem against arbitrary perturbations. The purpose here would be to study the evolution of the crack front toward "smoothness", or contrarily "disorder". This could be achieved by taking the Fourier transform of the perturbation so as to reduce the problem to the study of the evolution of sinusoidal perturbations, following the line just sketched. The previous study suggests that Fourier components of wavelength longer than λ_c will grow and the other ones decrease; that is, perturbations of short wavelength will disappear and only those of long wavelength will develop. But it is difficult to say a priori if the resulting crack front will become more "smooth" or more "disordered". Clearly, these ambiguous notions need to be given an accurate mathematical definition before any discussion is possible.

(3) The third one is related to non-linear effects disregarded in the first-order perturbation analysis. More specifically, the following problem arises. The critical wavelength evidenced here is proportional to the width of the crack. Thus, let us consider a wavy perturbation of the crack front of wavelength larger than the critical one. Then the amplitude of the oscillations will grow, but the width of the crack and therefore the critical wavelength will do just the same. Therefore the wavelength of the perturbation will become smaller than the critical one, and stability again prevail, after a certain distance of propagation. But if this distance is too large, the first-order perturbation method used in this paper may become invalid and non-linear effects important. It is improbable that this topic can be addressed analytically, but it may be handled using numerical methods (see for instance, Bower and Ortiz (1990) and Lazarus (1999)).

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